Exercise 60

If $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

$$\text{(a)} \ \lim_{x \to 0} f(x) \qquad\qquad \text{(b)} \ \lim_{x \to 0} \frac{f(x)}{x}$$

Solution

Plugging in $x = 0$ or applying the Quotient Law is out of the question because the denominator becomes zero. In order for this limit to exist and be equal to 5, the numerator must contain a factor of x^2 that cancels out with the one in the denominator. For simplicity, let

$$f(x) = Ax^2,$$

where A is a constant to be determined.

$$\lim_{x \to 0} \frac{f(x)}{x^2} = 5$$

$$\lim_{x \to 0} \frac{Ax^2}{x^2} = 5$$

$$\lim_{x \to 0} A = 5$$

$$A = 5$$

Substitute this value for A into the hypothesis.

$$f(x) = 5x^2$$

Therefore,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} 5x^2 = 5(0)^2 = 0,$$

and

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{5x^2}{x} = \lim_{x \to 0} 5x = 5(0) = 0.$$